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2002 J. Phys. A: Math. Gen. 35 6049

(http://iopscience.iop.org/0305-4470/35/29/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6049–6053 PII: S0305-4470(02)31967-X

The construction of space-like surfaces with $k_1k_2 - m(k_1 + k_2) = 1$ in Minkowski three-space

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Received 18 December 2001, in final form 20 March 2002
Published 12 July 2002
Online at stacks.iop.org/JPhysA/35/6049

Abstract

From solutions of the sinh-Laplace equation, we construct a family of space-like surfaces with $k_1k_2 - m(k_1 + k_2) = 1$ in Minkowski three-space, where k_1 and k_2 are principal curvatures and m is an arbitrary constant.

PACS numbers: 02.40.Vh, 02.30.Ik

From the point of view of soliton theory, for a surface in three-dimensional spaces, the Gauss–Weingarten formulae are a Lax pair of Gauss–Codazzi equations [1]. However, the Gauss–Codazzi equations have been obtained in [2] and [3] for any Weingarten surfaces, i.e., surfaces for which the principal curvatures k_1 and k_2 satisfy a functional relationship $f(k_1, k_2) = 0$. So, in theory, from a solution of the Gauss–Codazzi equations, by solving the Gauss–Weingarten formulae, one can construct a Weingarten surface whose principal curvatures k_1 and k_2 satisfy the given relationship $f(k_1, k_2) = 0$. But, in practice, for an arbitrary function $f(k_1, k_2) = 0$, the Gauss–Codazzi equations are nonlinear, and are, in general, difficult to solve. Even if a solution of the Gauss–Codazzi equations is given, one still faces the technical challenge of solving the Gauss–Codazzi formulae. For constant negative curvature space-like surfaces (i.e. $k_1k_2 = 1$) in Minkowski three-space M^3, the Gauss–Codazzi equation is the sinh-Laplace equation [4]. From solutions of the sinh-Laplace equation, Hu [5] has constructed a family of constant negative curvature space-like surfaces in M^3. In this paper, we first point out that for Weingarten space-like surfaces with $k_1k_2 - m(k_1 + k_2) = 1$ (m is an arbitrary constant) in M^3, the Gauss–Codazzi equation is still the sinh-Laplace equation. Then, by solving the Gauss–Weingarten formulae, we construct a family of space-like surfaces with $k_1k_2 - m(k_1 + k_2) = 1$, which are a generalization of surfaces obtained in [5]. We suppose that the surfaces discussed in this paper do not contain any umbilic points.

Let S be a space-like surface with $k_1k_2 - m(k_1 + k_2) = 1$, or equivalently, $(k_1 - m)(k_2 - m) = l^2 (l^2 - m^2 = 1)$. Let $\{r; e_1, e_2, e_3\}$ be an orthonormal Lorentzian frame field on S such that e_1, e_2 are tangent vector fields in the principal directions, and e_3 is the normal vector field $\left(e_1^2 = e_2^2 = -e_3^2 = 1\right)$. Then we have the moving equations

$$dr = \omega_1 e_1 + \omega_2 e_2 \tag{1}$$

$$\mathrm{d}e_i = \sum_{j=1}^{3} \omega_{ij} e_j \qquad i = 1, 2, 3 \tag{2}$$

where $\omega_{12} = -\omega_{21}, \omega_{13} = \omega_{31}, \omega_{23} = \omega_{32}$.

Suppose the first and second fundamental forms of S are respectively

$$\mathrm{I} = a^2\, \mathrm{d}u^2 + b^2\, \mathrm{d}v^2 \qquad \mathrm{II} = k_1 a^2\, \mathrm{d}u^2 + k_2 b^2\, \mathrm{d}v^2 \qquad (a, b > 0)$$

then we have

$$\begin{aligned} &\omega_1 = a\, \mathrm{d}u \quad \omega_2 = b\, \mathrm{d}v \qquad \omega_{13} = \omega_{31} = -k_1 a\, \mathrm{d}u \\ &\omega_{23} = \omega_{32} = -k_2 b\, \mathrm{d}v \qquad \omega_{12} = -\omega_{21} = -\frac{a_v}{b}\, \mathrm{d}u + \frac{b_u}{a}\, \mathrm{d}v \end{aligned} \tag{3}$$

and the Codazzi equations

$$(k_1 - k_2) a_v + k_{1v} a = 0 \qquad (k_2 - k_1) b_u + k_{2u} b = 0. \tag{4}$$

Since $(k_1 - m)(k_2 - m) = l^2$ $(l^2 - m^2 = 1)$, we can assume that

$$k_1 - m = l \frac{l \sinh \frac{\alpha}{2} - m \cosh \frac{\alpha}{2}}{l \cosh \frac{\alpha}{2} - m \sinh \frac{\alpha}{2}} \qquad k_2 - m = l \frac{l \cosh \frac{\alpha}{2} - m \sinh \frac{\alpha}{2}}{l \sinh \frac{\alpha}{2} - m \cosh \frac{\alpha}{2}}. \tag{5}$$

Then, using the Codazzi equations, we can choose parameters u and v such that

$$a = l \cosh \frac{\alpha}{2} - m \sinh \frac{\alpha}{2} \qquad b = l \sinh \frac{\alpha}{2} - m \cosh \frac{\alpha}{2}.$$

Hence

$$\begin{aligned} &\omega_1 = \left(l \cosh \frac{\alpha}{2} - m \sinh \frac{\alpha}{2} \right) \mathrm{d}u \qquad \omega_2 = \left(l \sinh \frac{\alpha}{2} - m \cosh \frac{\alpha}{2} \right) \mathrm{d}v \\ &\omega_{12} = -\omega_{21} = -\frac{\alpha_v}{2}\, \mathrm{d}u + \frac{\alpha_u}{2}\, \mathrm{d}v \\ &\omega_{13} = \omega_{31} = \sinh \frac{\alpha}{2}\, \mathrm{d}u \qquad \omega_{23} = \omega_{32} = \cosh \frac{\alpha}{2}\, \mathrm{d}v. \end{aligned} \tag{6}$$

From equation (6), equation (2) can be rewritten as

$$\begin{aligned} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_u &= \begin{pmatrix} 0 & -\frac{1}{2}\alpha_v & \sinh \frac{\alpha}{2} \\ \frac{1}{2}\alpha_v & 0 & 0 \\ \sinh \frac{\alpha}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_v &= \begin{pmatrix} 0 & \frac{1}{2}\alpha_u & 0 \\ -\frac{1}{2}\alpha_u & 0 & \cosh \frac{\alpha}{2} \\ 0 & \cosh \frac{\alpha}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \end{aligned} \tag{7}$$

The Gauss equation is the sinh-Laplace equation

$$\alpha_{uu} + \alpha_{vv} = \sinh \alpha \tag{8}$$

and equation (7) are a Lax pair of equation (8).

The sinh-Laplace equation (8) has the following solutions

$$\alpha = 2 \sinh^{-1} \left(-\frac{1}{\sinh(\lambda u + \mu v)} \right) \tag{9}$$

where λ and μ are constants and satisfy $\lambda^2 + \mu^2 = 1$, $\lambda > 0$, $\lambda u + \mu v < 0$. Then

$$\sinh \frac{\alpha}{2} = -\frac{1}{\sinh(\lambda u + \mu v)} \qquad \cosh \frac{\alpha}{2} = -\frac{\cosh(\lambda u + \mu v)}{\sinh(\lambda u + \mu v)}. \tag{10}$$

Now the first part of the Lax pair (7) can be written as

$$e_{1u} = -\mu \sinh\frac{\alpha}{2}e_2 + \sinh\frac{\alpha}{2}e_3 \qquad e_{2u} = \mu \sinh\frac{\alpha}{2}e_1 \qquad e_{3u} = \sinh\frac{\alpha}{2}e_1. \tag{11}$$

Letting

$$a = -e_2 + \mu e_3 \qquad b = \lambda e_1 - \mu e_2 + e_3 \qquad c = \lambda e_1 + \mu e_2 - e_3 \tag{12}$$

then equation (11) is equivalent to

$$a_u = 0 \qquad b_u = \lambda \sinh\frac{\alpha}{2}b \qquad c_u = -\lambda \sinh\frac{\alpha}{2}c. \tag{13}$$

Integrating equation (13) we have

$$a = a_0(v) \qquad b = -b_0(v)\coth\frac{\lambda u + \mu v}{2} \qquad c = -c_0(v)\tanh\frac{\lambda u + \mu v}{2} \tag{14}$$

where $a_0(v)$, $b_0(v)$ and $c_0(v)$ are vector-valued functions of v.

Substituting equation (14) into the second part of the Lax pair (7), through complicated calculations we have

$$a_0'(v) = -\tfrac{1}{2}b_0(v) + \tfrac{1}{2}c_0(v) \qquad b_0'(v) = -a_0(v) \qquad c_0'(v) = a_0(v). \tag{15}$$

The general solutions of equation (15) are

$$\begin{aligned} a_0(v) &= c_1\cosh v + c_2\sinh v \\ b_0(v) &= -c_1\sinh v - c_2\cosh v + c_3 \\ c_0(v) &= c_1\sinh v + c_2\cosh v + c_3 \end{aligned} \tag{16}$$

where c_1, c_2 and c_3 are arbitrary constant vectors.

Let $v = 0$ and $u \to -\infty$. From equations (14) and (16), we have

$$a_0(0) = c_1 \qquad b_0(0) = -c_2 + c_3 \qquad c_0(0) = c_2 + c_3. \tag{17}$$

Considering equation (12) we have

$$a_0(0) = -e_2^0 + \mu e_3^0 \qquad b_0(0) = \lambda e_1^0 - \mu e_2^0 + e_3^0 \qquad c_0(0) = \lambda e_1^0 + \mu e_2^0 - e_3^0 \tag{18}$$

where $\{e_1^0, e_2^0, e_3^0\}$ form an orthonormal Lorentzian basis of M^3. From equation (12), we obtain the general solutions of the Lax pair (7)

$$\begin{aligned} e_1 &= -\coth\xi\, e_1^0 - \frac{1}{\lambda\sinh\xi}(\sinh v - \mu\cosh v)e_2^0 - \frac{1}{\lambda\sinh\xi}(\cosh v - \mu\sinh v)e_3^0 \\ e_2 &= -\frac{\mu}{\lambda\sinh\xi}e_1^0 + \frac{1}{\lambda^2}(\cosh v - \mu\sinh v - \mu\coth\xi(\sinh v - \mu\cosh v))e_2^0 \\ &\quad + \frac{1}{\lambda^2}(\sinh v - \mu\cosh v - \mu\coth\xi(\cosh v - \mu\sinh v))\, e_3^0 \end{aligned} \tag{19}$$

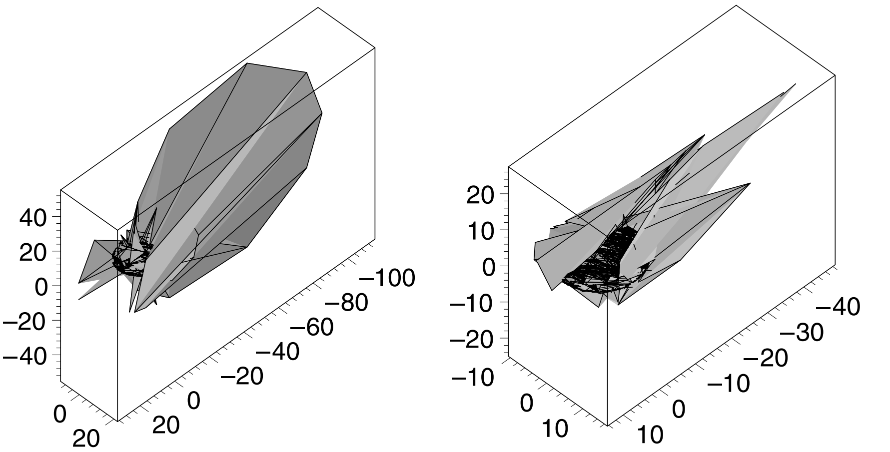
where $\xi = \lambda u + \mu v$.

Now we solve equation (1), i.e.

$$\mathrm{d}r = \left(l\cosh\frac{\alpha}{2} - m\sinh\frac{\alpha}{2}\right)\mathrm{d}u\, e_1 + \left(l\sinh\frac{\alpha}{2} - m\cosh\frac{\alpha}{2}\right)\mathrm{d}v\, e_2. \tag{20}$$

From

$$r_u = \left(l\cosh\frac{\alpha}{2} - m\sinh\frac{\alpha}{2}\right)e_1 \tag{21}$$



(*a*) (*b*)

Figure 1. Space-like surfaces whose principal curvatures k_1 and k_2 satisfy $k_1k_2 - (k_1 + k_2) = 1$.

and by using equations (10) and (19), we have

$$\begin{aligned} r = {} & r_0(v) + \frac{1}{\lambda}\left(l(\xi - \coth\xi) + \frac{m}{\sinh\xi}\right)e_1^0 \\ & \frac{1}{\lambda^2}\left(\frac{1}{\sinh\xi}(\mu\cosh v - \sinh v) + m(\sinh v - \mu\cosh v)\cot\xi\right)e_2^0 \\ & \frac{1}{\lambda^2}\left(\frac{1}{\sinh\xi}(\mu\sinh v - \cosh v) + m(\cosh v - \mu\sinh v)\cot\xi\right)e_3^0 \end{aligned} \tag{22}$$

where $r_0(v)$ is a vector-valued function of v.

From

$$r_v = \left(l\sinh\frac{\alpha}{2} - m\cosh\frac{\alpha}{2}\right)e_2 \tag{23}$$

and by using equations (10) and (19), we have

$$r_0'(v) + \frac{l\mu}{\lambda}e_1^0 + \frac{m\mu}{\lambda^2}(\sinh v - \mu\cosh v)e_2^0 + \frac{m\mu}{\lambda^2}(\cosh v - \mu\sinh v)e_3^0 = 0 \tag{24}$$

therefore

$$r_0(v) = -\frac{l\mu}{\lambda}ve_1^0 - \frac{m\mu}{\lambda^2}(\cosh v - \mu\sinh v)e_2^0 - \frac{m\mu}{\lambda^2}(\sinh v - \mu\cosh v)e_3^0. \tag{25}$$

From equations (22) and (25), we obtain general solutions of equation (1)

$$\begin{aligned} r = {} & \left(lu - \frac{l}{\lambda}\coth\xi + \frac{m}{\lambda\sinh\xi}\right)e_1^0 + \left(-\frac{m\mu}{\lambda^2}(\cosh v - \mu\sinh v)\right. \\ & \left. - \frac{1}{\lambda^2\sinh\xi}(\sinh v - \mu\cosh v) + \frac{m}{\lambda^2}(\sinh v - \mu\cosh v)\coth\xi\right)e_2^0 \\ & + \left(-\frac{m\mu}{\lambda^2}(\sinh v - \mu\cosh v) - \frac{1}{\lambda^2\sinh\xi}(\cosh v - \mu\sinh v)\right. \\ & \left. + \frac{m}{\lambda^2}(\cosh v - \mu\sinh v)\coth\xi\right)e_3^0. \end{aligned} \tag{26}$$

Without loss of generality, we can choose an orthonormal Lorentzian basis of M^3

$$\left(\tilde{e}_1^0, \tilde{e}_2^0, \tilde{e}_3^0\right) = \left(e_1^0, \frac{1}{\lambda}\left(e_2^0 - \mu e_3^0\right), \frac{1}{\lambda}\left(e_3^0 - \mu e_2^0\right)\right) \tag{27}$$

then we obtain a family of space-like surfaces with $k_1k_2 - m(k_1 + k_2) = 1$ in M^3

$$S_{\lambda,m} := \quad \begin{aligned} x_1(u, v) &= lu - \frac{l}{\lambda}\coth(\lambda u + \mu v) + \frac{m}{\lambda \sinh(\lambda u + \mu v)} \\ x_2(u, v) &= -\frac{l \sinh v}{\lambda \sinh(\lambda u + \mu v)} - \frac{m\mu \cosh v}{\lambda} + \frac{m}{\lambda} \sinh v \coth(\lambda u + \mu v) \\ x_3(u, v) &= -\frac{l \cosh v}{\lambda \sinh(\lambda u + \mu v)} - \frac{m\mu \sinh v}{\lambda} + \frac{m}{\lambda} \cosh v \coth(\lambda u + \mu v) \end{aligned} \tag{28}$$

where $\lambda^2 + \mu^2 = 1$ and $l^2 - m^2 = 1$.

If $l = 1, m = 0$, then $S_{\lambda,0}$ are the space-like surfaces S_λ obtained in [1].

By using the software MAPLE, one can plot $S_{\lambda,\mu}$ for certain values of the parameters l, m, λ and μ. For example, the shape of $S_{1,1}$ is shown in figure 1(*a*) when $l = \sqrt{2}, m = 1, \lambda = 1$ and $\mu = 0$. The shape of $S_{\frac{\sqrt{2}}{2},1}$ is shown in figure 1(*b*) when we let $l = \sqrt{2}, m = 1$ and $\lambda = \mu = \frac{\sqrt{2}}{2}$.

Acknowledgment

This work is supported by the National Natural Science Foundation of China through the grant no 10226037.

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